Fisher Information and Prony Representability of Fractional Relaxation Models

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Abstract. Relaxation spectra encode complex material and circuit dynamics. A central problem is determining when continuous spectrum models can be accurately represented by finite physical networks. This work establishes a correspondence between the Mellin transform of a relaxation spectrum and its Fisher information metric, enabling a rigorous classification of these models. We distinguish between the "finite Prony class," representable by limited exponential sums, and the "transcendental class" (e.g., log-normal, Mittag-Leffler, etc.), which requires infinite Prony approximations. Analyzing specific spectra shows that the required number of Prony elements scales differently depending on the distribution (e.g. square root of log error for log-normal). The resulting Fisher information metric serves as a fundamental complexity measure with direct applications in viscoelastic constitutive laws and fractional-order control design.

Keywords: Mellin transform, Fisher information, Prony series, fractional Maxwell model, relaxation spectrum, viscoelasticity

1 Introduction

Relaxation phenomena are ubiquitous in nature. In the small signal/small response perspective relaxation is linearized around the unperturbed state. Examples can be given by stress decay in polymers, dielectric response in glasses, and electric response in resistor-capacitor networks etc. The relaxation process can be described by a distribution of relaxation times – its relaxation spectrum. From an empirical perspective the relaxation spectrum can be described approximately by series of decaying exponents – a finite Prony series [8]. The central question is then whether such a description is faithful, i.e. if such a spectrum admits a finite Prony representation and when it does requires an infinite ladder of such relaxations.

The present work establishes an exact correspondence between the Mellin transform of a relaxation spectrum and the Fisher information metric on the statistical manifold of log-relaxation times. We establish the so-called Mellin-fisher identity which results in a universal approximation criterion given certain uniform error. Two concrete examples are presented. For the log-normal

spectrum – the maximum-entropy relaxation distribution — the required Prony modes scale as $O(\sigma\sqrt{\ln(1/\varepsilon)})$, with the Fisher information σ^2 controlling computational complexity. For the Mittag-Leffler (fractional Maxwell) spectrum, the scaling is quadratic logarithmic: $O(\ln^2(1/\varepsilon)/[\alpha(1-\alpha)])$, a direct consequence of its heavier exponential tails.

2 Mathematical Background

For a linear viscoelastic material, the relaxation modulus $G(t)$ and the relaxation spectrum $H(\tau) \geq 0$ are related by

$$G(t) = G_\infty + \int_0^\infty e^{-t/\tau}\, H(\tau)\, \frac{d\tau}{\tau} \tag{1}$$

2.1 Information-Geometric Background

Definition 1 (Fisher metric). *For a one-parameter family of probability densities $P(t \mid \theta)$, the Fisher metric is defined by*

$$I(\theta) = \int_0^\infty P(t \mid \theta) \left(\frac{\partial}{\partial\theta} \log P(t \mid \theta) \right)^2 dt.$$

Uneyama [10] interpreted the normalized Maxwell kernel $P_M(t|\tau) = \tau^{-1}e^{-t/\tau}$ as a probability density and computed its Fisher information $I(\tau) = 1/\tau^2$. Under the reparameterization $\xi = \ln(\tau/\tau_0)$, the Fisher metric becomes constant, $I(\xi) = 1$. This shows that the log-time coordinate is the natural affine coordinate for the relaxation-time manifold.

Applying the maximum entropy principle under fixed variance in the ξ variable yields the log-normal relaxation spectrum:

$$H_{\mathrm{LN}}(\tau) = \frac{G_0}{\sqrt{2\pi\sigma^2}} \exp\!\left(-\frac{(\ln\tau - \ln\tau_0)^2}{2\sigma^2} \right), \tag{2}$$

which is the most probable spectrum in the absence of further information.

2.2 The Mellin–Prony Framework

The Mellin-space analysis begins from the constitutive equation in the frequency domain:

$$G^*(\omega) = G_\infty + \int_0^\infty \frac{i\omega\tau}{1 + i\omega\tau} H(\tau) \frac{d\tau}{\tau}. \tag{3}$$

Applying the Mellin transform and the convolution theorem yields the factorization [7]:

$$\tilde{G}(s) = K(s)\tilde{H}(-s), \qquad K(s) = \pi e^{-i\pi s/2} \csc(\pi s). \tag{4}$$

The kernel $K(s)$ has poles at all integers. It was demonstrated that a spectrum belongs to the finite Prony class $\mathcal{P}$ if and only if its pole lattice aligns with the integers and residues decouple (Theorem 1 of [7]).

Applying this result to Uneyama's example proceeds as follows. The log-normal Mellin transform,

$$\tilde{H}_{\mathrm{LN}}(s) = \exp\Big(\mu s + \tfrac{\sigma^2}{2} s^2\Big), \quad \mu = \ln \tau_0 \tag{5}$$

is entire of order 2 – hence **transcendental**, requiring an infinite Prony ladder.

2.3 Useful definitions

Definition 2 (Log-time density and two-sided Laplace transform). *Let $H(\tau)$ be a relaxation spectrum with finite spectral strength*

$$Z = \int_0^\infty H(\tau)\, \frac{d\tau}{\tau} < \infty.$$

Define the logarithmic time variable $u = \ln(\tau/\tau_0)$, where τ_0 is a characteristic relaxation time. The corresponding normalized probability density on $\mathbb{R}$ is

$$p(u) := \frac{1}{Z} H(\tau_0 e^u). \tag{6}$$

Assume that $p(u)$ has an open strip of convergence for its two-sided Laplace transform, i.e., there exists an interval $\Sigma = (\alpha, \beta) \subset \mathbb{R}$ containing 0. Then for any complex $s = \sigma + i\xi$ in the strip $\sigma \in \Sigma$, the Mellin transform of H converges absolutely and defines an analytic function. Under the normalization condition, this Mellin transform coincides exactly with the two-sided Laplace transform (moment generating function) of p:

$$\tilde{H}(s) = \int_0^\infty H(\tau)\tau^{s-1} d\tau = \int_{\mathbb{R}} p(u) e^{su} du =: Z\tau_0^s M(s). \tag{7}$$

Thus $\tilde{H}(s)$ is analytic on the vertical strip $\{s \in \mathbb{C} : \mathrm{Re}(s) \in \Sigma\}$.

Definition 3 (Exponentially tilted family). *For $s \in \Sigma$, define*

$$p(u; s) := \frac{p(u) e^{su}}{M(s)}, \qquad M(s) = \int_{\mathbb{R}} p(u) e^{su}\, du.$$

This is a regular one-parameter exponential family with sufficient statistic u.

3 The Central Mellin–Fisher Identity

The central theoretical result of the paper can be summarized in the following Theorem.

Theorem 1 (Mellin–Fisher identity). *For the exponentially tilted family $p(u;s)$ with $s \in \Sigma$, the Fisher information with respect to the real parameter s is*

$$g(s) := \int_{\mathbb{R}} p(u;s) \left(\frac{\partial}{\partial s} \log p(u;s) \right)^2 du. \tag{8}$$

Thus the Fisher information satisfies

$$g(s) = \mathrm{Var}_{p(\cdot;s)}[u] = \frac{d^2}{ds^2} \log M(s) = \frac{d^2}{ds^2} \log \tilde{H}(s). \tag{9}$$

In particular, at $s = 0$,

$$\left. \frac{d^2}{ds^2} \log \tilde{H}(s) \right|_{s=0} = \mathrm{Var}_p[u] = g(0). \tag{10}$$

Proof. For the tilted family $p(u;s) = p(u)e^{su}/M(s)$, the score is

$$\frac{\partial}{\partial s} \log p(u;s) = u - \frac{M'(s)}{M(s)} = u - \mathbb{E}_{p(\cdot;s)}[u]. \tag{11}$$

Hence

$$g(s) = \int p(u;s) \left(u - \mathbb{E}_{p(\cdot;s)}[u] \right)^2 du = \mathrm{Var}_{p(\cdot;s)}[u]. \tag{12}$$

Differentiating $\log M(s)$ gives

$$\frac{d}{ds} \log M(s) = \mathbb{E}_{p(\cdot;s)}[u], \tag{13}$$

$$\frac{d^2}{ds^2} \log M(s) = \mathrm{Var}_{p(\cdot;s)}[u] = g(s). \tag{14}$$

By Definition 1, $\tilde{H}(s) = Z\, \tau_0^s M(s)$, so

$$\log \tilde{H}(s) = \log Z + s \log \tau_0 + \log M(s),$$

and the second derivative of the first two terms is zero. Therefore,

$$\frac{d^2}{ds^2} \log \tilde{H}(s) = \frac{d^2}{ds^2} \log M(s) = g(s).$$

For the log-normal spectrum, $p(u) = \mathcal{N}(\mu, \sigma^2)$, so $g(0) = \sigma^2$. The curvature of the log-Mellin transform at $s = 0$ directly encodes the spectral variance.

4 Prony Approximation of the Log-Normal Spectrum

Theorem 2 (Log-normal Prony bound). *Let $p(u) = (2\pi\sigma^2)^{-1/2} e^{-u^2/(2\sigma^2)}$. For any $\varepsilon \in (0, 1/2)$, there exists a finite Prony series with*

$$N = 2M + 1 \geq \frac{4\sigma}{\pi} \sqrt{2 \ln(4/\varepsilon) \ln(4C_\sigma/\varepsilon)} \tag{15}$$

geometrically spaced terms ($\tau_k = e^{k\Delta}$, $\Delta = \pi\sigma/\sqrt{\ln(4C_\sigma/\varepsilon)}$) such that $\sup_{t>0} |G(t) - G_N(t)| \leq \varepsilon$.

Proof (Proof sketch). **Discretization:** By Poisson summation and Paley–Wiener estimates, $|\widehat{f}_t(\xi)| \le C_\sigma e^{-\sigma^2\xi^2/4}$, so the aliasing error $\le \varepsilon/2$ for $\Delta \le \pi\sigma/\sqrt{\ln(4C_\sigma/\varepsilon)}$.

Truncation: The Gaussian tail gives $\sum_{|k|>M} f_t(k\Delta)\Delta \le 2Q(M\Delta/\sigma) \le \varepsilon/2$ for $M = \lceil(\sigma/\Delta)\sqrt{2\ln(4/\varepsilon)}\rceil$.

Mode count: $N \approx \frac{2\sqrt{2}}{\pi}\sqrt{\ln(4/\varepsilon)\ln(4C_\sigma/\varepsilon)}$, confirming $N = O(\sigma\sqrt{\ln(1/\varepsilon)})$.

5 Fisher Analysis of the Mittag-Leffler Spectrum

An important class of relaxation spectra are given by the fractional Maxwell model[4]. The relaxation modulus for the fractional Maxwell model with fractional order $\alpha \in (0,1)$, characteristic time τ_0, and modulus scale G_0 is

$$G(t) = G_0\, E_\alpha\big(-(t/\tau_0)^\alpha\big), \tag{16}$$

where the one-parameter Mittag-Leffler function is defined by the power series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \tag{17}$$

and is an entire function of z for $\alpha > 0$. Its Laplace transform admits the simple rational form in the Laplace variable p:

$$\widehat{G}(p) = \int_0^\infty e^{-pt}\, G(t)\, dt = G_0\, \frac{p^{\alpha-1}}{\tau_0^{-\alpha} + p^\alpha}, \qquad \mathrm{Re}(p) > 0. \tag{18}$$

The normalized log-time density is:

$$p_\alpha(u) = \frac{\sin(\pi\alpha)}{2\pi[\cosh(\alpha u) + \cos(\pi\alpha)]}, \qquad u \in \mathbb{R}. \tag{19}$$

The limiting cases are $p_1(u) = \delta(u)$ (single Maxwell mode) and $p_0(u) \to$ uniform (power-law, scale-free).

5.1 Mellin Transform and Fisher information

The Mellin transform of the normalized Mittag-Leffler spectrum is [5]:

$$\tilde{H}_\alpha(s) = \frac{\Gamma(s/\alpha)\,\Gamma(1 - s/\alpha)}{\Gamma(s)\,\Gamma(1-s)} = \frac{\sin(\pi s)}{\sin(\pi s/\alpha)}, \qquad 0 < \mathrm{Re}(s) < \alpha. \tag{20}$$

The poles at $s \in \{-\alpha k\} \cup \{\alpha(1+k)\}$ are incommensurate with the integer poles of $\Gamma(s)\Gamma(1-s)$ for $\alpha \notin \mathbb{Q}$, and the residue decoupling condition fails for all $\alpha \in (0,1)$. Hence the Mittag-Leffler spectrum is **transcendental** for all $\alpha \in (0,1)$. Applying the Mellin–Fisher identity yields:

$$\frac{d^2}{ds^2}\log\tilde{H}_\alpha(s) = \frac{\pi^2}{\alpha^2\sin^2(\pi s/\alpha)} - \frac{\pi^2}{\sin^2(\pi s)}. \tag{21}$$

Laurent expansion around $s = 0$ cancels the $1/s^2$ poles, yielding:

$$g_\alpha(0) = \frac{\pi^2}{3}\left(\frac{1}{\alpha^2} - 1\right) \tag{22}$$

5.2 Prony Approximation Bound

Theorem 3 (Mittag-Leffler Prony bound). *For $\alpha \in (0,1)$ and $\varepsilon \in (0,1/2)$, there exists a finite Prony series with spacing $\Delta = \pi(1-\alpha)/\ln(4/\varepsilon)$ such that $\sup_{t>0} |G(t) - G_N(t)| \leq \varepsilon$, with mode count:*

$$N \sim \frac{2}{\pi\alpha(1-\alpha)} \ln^2(1/\varepsilon) \quad \text{as } \varepsilon \to 0 \tag{23}$$

Proof (Proof sketch). **Fourier decay:** The characteristic function satisfies $|\widehat{f}_t(\xi)| \leq C_\alpha e^{-\pi|\xi|(1/\alpha-1)}$, so Poisson aliasing $\leq \varepsilon/2$ requires $\Delta \leq \pi(1-\alpha)/\ln(4/\varepsilon)$.

Truncation: Since $p_\alpha(u) \lesssim e^{-\alpha|u|}$, the tail $\leq \varepsilon/2$ for $M = \lceil \ln(4/\varepsilon)/(\alpha\Delta) \rceil$.

Mode count: $N = 2M + 1$, and substituting Δ gives $M = \frac{\ln^2(4/\varepsilon)}{\pi\alpha(1-\alpha)}$, confirming $N = O(\ln^2(1/\varepsilon))$.

The contrast with the log-normal case is profound: Gaussian tails yield $\Delta \propto 1/\sqrt{\ln(1/\varepsilon)}$; exponential tails force $\Delta \propto 1/\ln(1/\varepsilon)$, and the product of this finer spacing with the same truncation range yields the log-squared penalty.

For the Mittag-Leffler kernel $G(t) = E_{1/2}(-\sqrt{t}) = e^t \operatorname{erfc}(\sqrt{t})$, Figure 1 shows that a ladder with $N = 100$ rungs and $q \approx 1.3$ achieves a relative L^∞ error below 10^{-8} on $t \in [10^{-2}, 10^2]$. This covers four decades of time with a cost equivalent to a 201-term Prony series – competitive with, the rational Padé approximants used in practice [1], while being derived analytically from the spectral density $H_c(\tau)$ rather than fitted to data. The ladder construction used here is equivalent to trapezoidal quadrature on the log-time axis, in the spirit of Beylkin and Monzón [3, 2]. The present framework goes further by identifying Fisher information as the relevant complexity parameter and by distinguishing between spectra that admit finite Prony representations and those that are intrinsically infinite-dimensional.

6 Discussion

The Mellin–Fisher identity gives a principled way to measure the complexity of a relaxation spectrum. The Fisher information of the log-time density quantifies the effective spectral width of the relaxation kernel. In this sense, Theorems 2 and 3 express approximation cost in terms of an intrinsic information-geometric scale: broader spectra require more Prony modes to reach a prescribed tolerance. For the log-normal case, the Gaussian decay in log-time leads to a square-root logarithmic growth of the mode count, whereas the Mittag–Leffler spectrum, with heavier tails, incurs a stronger logarithmic penalty. Thus the Fisher metric provides the natural bridge between spectral width and finite-dimensional approximability.

Presented results are directly relevant for numerical modeling of viscoelastic and anomalously diffusive systems. In finite-element and boundary-element implementations, Prony ladders replace hereditary integrals with a finite set of internal variables, so the computational cost grows only linearly with the number of ladder terms rather than with the full loading history [9].

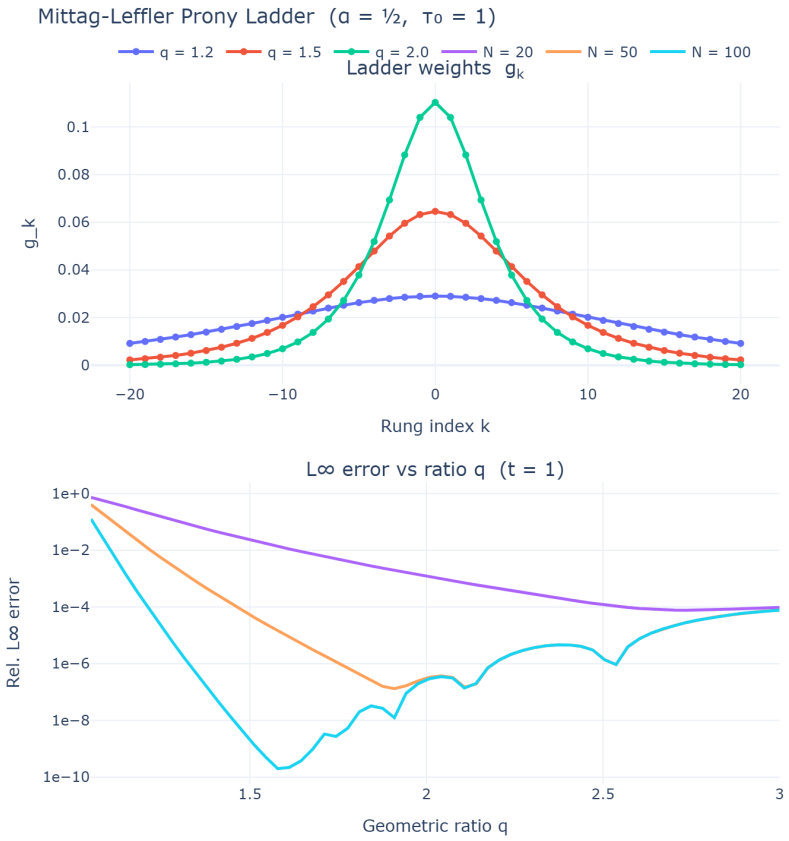


Fig. 1. Mittag-Leffler Prony ladder ($\alpha = \frac{1}{2}$): *Top:* ladder weights $g_k = H_c(\tau_k)\ln q$ vs. rung index k for $q \in \{1.2, 1.5, 2.0\}$. *Bottom:* L^∞ relative error vs. geometric ratio q at $t = 1$ for $N \in \{20, 50, 100\}$ rungs.

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Appendix

A Mellin transform

For a suitable function $f(t)$ the Mellin transform is defined by the complex-valued integral

$$\mathcal{M}\{f\}(s) \equiv \tilde{f}(s) := \int_0^\infty t^{\,s-1} f(t)\, dt,$$

whenever the integral converges absolutely in some vertical strip $a < \Re s < b$ of the complex plane [6].

B Relation between Laplace transform and $H(\tau)$

For the fractional Maxwell model, $G_\infty = 0$. Taking the Laplace transform and exchanging integrals (justified by absolute convergence for $\mathrm{Re}(p) > 0$):

$$\widehat{G}(p) = \int_0^\infty \frac{H(\tau)}{p + 1/\tau}\,\frac{d\tau}{\tau} = \int_0^\infty \frac{H(\tau)}{p\tau + 1}\,d\tau. \tag{24}$$

B.1 Stieltjes inversion

Equation (24) is a Stieltjes transform representation. The spectrum $H(\tau)$ can be recovered by the Stieltjes–Perron inversion formula. Let $p = -1/\tau + i\varepsilon$ and take the limit $\varepsilon \to 0^+$. The standard inversion identity

$$\lim_{\varepsilon\to 0^+} \frac{1}{\pi}\,\mathrm{Im}\,\widehat{G}\left(-\frac{1}{\tau} + i\varepsilon\right) = \frac{H(\tau)}{\tau} \tag{25}$$

holds for distributions $H(\tau)$ that are locally integrable with respect to $d\tau/\tau$.

Applying (25) to the Laplace transform (18) with $G_0 = 1$:

$$\widehat{G}(p) = \frac{p^{\alpha-1}}{\tau_0^{-\alpha} + p^\alpha}. \tag{26}$$

Set $p = -1/\tau + i\varepsilon = e^{\pm i\pi}/\tau$. In the limit $\varepsilon \to 0^+$, the term p^α on the branch cut becomes

$$p^\alpha \to \tau^{-\alpha} e^{\pm i\pi\alpha} = \tau^{-\alpha}\big(\cos(\pi\alpha) \pm i\sin(\pi\alpha)\big), \tag{27}$$

and $p^{\alpha-1} = p^\alpha/p$.

Taking the imaginary part (after rationalizing the denominator) yields

$$H(\tau)\tau = \frac{1}{\pi}\,\mathrm{Im}\,\widehat{G}\left(-\frac{1}{\tau} + i0^+\right) = \frac{1}{\pi}\,\frac{\tau_0^{-\alpha}\tau^{1-\alpha}\sin(\pi\alpha)}{\tau_0^{-2\alpha} + 2\tau_0^{-\alpha}\tau^{-\alpha}\cos(\pi\alpha) + \tau^{-2\alpha}}. \tag{28}$$

Simplifying, the relaxation spectrum in τ is

$$H(\tau) = \frac{1}{\pi}\,\frac{(\tau/\tau_0)^\alpha \sin(\pi\alpha)}{1 + 2(\tau/\tau_0)^\alpha\cos(\pi\alpha) + (\tau/\tau_0)^{2\alpha}}. \tag{29}$$

B.2 Transition to log-time density

Introduce the logarithmic variable $u = \ln(\tau/\tau_0)$, so that $\tau = \tau_0 e^u$ and $d\tau/\tau = du$. The spectrum transforms as a density according to

$$H(\tau)\,\frac{d\tau}{\tau} = \widehat{H}(u)\,du, \tag{30}$$

hence the unnormalized log-time density is $\widehat{H}(u) = H(\tau_0 e^u)$. From (29):

$$\widehat{H}(u) = \frac{1}{\pi}\,\frac{e^{\alpha u}\sin(\pi\alpha)}{1 + 2e^{\alpha u}\cos(\pi\alpha) + e^{2\alpha u}}. \tag{31}$$

Using the hyperbolic identity $\cosh(\alpha u) = (e^{\alpha u} + e^{-\alpha u})/2$, the denominator can be rewritten as

$$1 + 2e^{\alpha u}\cos(\pi\alpha) + e^{2\alpha u} = 2e^{\alpha u}\big[\cosh(\alpha u) + \cos(\pi\alpha)\big] \tag{32}$$

The factor $e^{\alpha u}$ cancels with the numerator, giving the hyperbolic-trigonometric form

$$\widehat{H}(u) = \frac{\sin(\pi\alpha)}{2\pi\big[\cosh(\alpha u) + \cos(\pi\alpha)\big]}. \tag{33}$$

This function is already normalized and we identify the probability density $p_\alpha(u) = \widehat{H}(u)$.

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